A Fekete-Szegö Inequality with Classes of Analytic Function Along with Its Subclasses , Extremals and Singularaties

S. K. Gandhi[1],Gurmeet Singh[2] ,*Preeti kumawat[3],G.S. Rathore[4],Lokendra kumawat[5]
[1,3,4,5] *Department of Mathematics and Statistics,Mohanlal Sukhadia University Udaipur, Rajasthan , 313001,India*
[1]*Email: gandhisk28@gmail.com*,
[3]*Email : preeti.kumawat30@gmail.com*,
[4]*Email: ganshyamsrathore@yahoo.co.in*,
[5]*Email : lokendrakumawat@yahoo.co.in*
[2] *Department of Mathematics, GSSDGS Khalsa College, Patiala, India*
[2]*Email: meetgur111@gmail.com*

Abstract

In this Paper we have introduced a Fekete-Szegö inequality with classes of analytic functions along with its subclasses extremals and Singularities by using principle of subordination and as so obtained sharp upper Bound of the function.

$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ belonging to these classes are also obtained.

Keywords : Bounded functions,Fekete-Szegö inequality,convex function, extremal function, Starlike functions, Inverse Starlike functions, Univalent functions.

1. Introduction

Let $\mathcal{A}$ denotes the class of the function of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1)$$

Which are analytic function in the unit disc $\mathbb{E} = \{z: |z| < 1|\}$,

Let $\mathcal{S}$ be the class of the functions of the form (1) which are analytic univalent in $\mathbb{E}$. Bieber Bach[8] proved that $|a_2| \leq 2$ for the functions $f(z) \in \mathcal{S}$. And Löwner[5] proved that $|a_3| \leq 3$ for the functions $f(z) \in \mathcal{S}$.. With the known estimates this inequality plays an important role to determining estimates of higher coefficients for some sub classes of $\boldsymbol{\mathcal{S}}$. {Chhichra[11], Babalola[6]}.

Using Löwner's method[5], Fekete and szego investigated a well known relation between $\boldsymbol{a_3}$ and $\boldsymbol{a_2^2}$ for the class

$|\boldsymbol{a_3} - \mu \boldsymbol{a_2^2}| \leq$

$$\begin{cases} 3 - 4\mu & ,if\ \mu \leq 0 \\ 1 + 2e^{\left(\frac{-2\mu}{1-\mu}\right)} & ,if\ 0 \leq \mu \leq 1 \\ 4\mu - 3 & ,if\ \mu \geq 1 \end{cases} \quad (2)$$

The Fekete–Szegő inequality is an inequality for the coefficients of univalent analytic functions found by Fekete and Szegő[10] , related to the Bieberbach conjecture. Finding similar estimates for other classes of functions is called the Fekete–Szegő problem.

Let $\mathbf{S}^*$ be the subclass of $\boldsymbol{\mathcal{S}}$ of univalent convex functions $\boldsymbol{h(z) = z + \sum_{n=2}^{\infty} c_n z^n} \in \boldsymbol{\mathcal{A}}$ satisfying the condition

$$\boldsymbol{Re}\frac{((\boldsymbol{zh'(z)}))}{\boldsymbol{h'(z)}} > 0, \boldsymbol{z} \in \mathbb{E}.$$
(3)

We are aware that a function $\boldsymbol{f(z)} \in \boldsymbol{\mathcal{A}}$ is said to be close to convex if there exist $\boldsymbol{g(z)} \in \mathbf{S}^*$ such that

$$\boldsymbol{Re}\frac{((\boldsymbol{zf'(z)}))}{\boldsymbol{g(z)}} > 0, \boldsymbol{z} \in \mathbb{E}. \quad (4)$$

Kaplan[18] proved that close to convex functions are univalent.

$$\mathrm{S}^*(\mathrm{A,B}) = \{\boldsymbol{f(z)} \in \boldsymbol{\mathcal{A}}\ ;\ \frac{((\boldsymbol{zf'(z)}))}{\boldsymbol{g(z)}} \prec \frac{\mathbf{1+Az}}{\mathbf{1+Bz}}, \boldsymbol{-1 \leq B \leq A \leq 1, z} \in \mathbb{E}\} \quad (5)$$

Where $\mathrm{S}^*(\mathrm{A,B})$ is a subclass of S^*.

Fekete-Szegö problem was studied by Abedel-Gawad[4] in the context of alpha quasi-convex function. Goel and Mehrok[13], Al-Shaqsi and Darus[1], Hayami and Owa[17], Al-Abbadi and Darus[9] have investigated the upper bound of $|\boldsymbol{a_3} - \mu \boldsymbol{a_2^2}|$for different functions in the class S.

And Gurmeet singh et al.[3] also introduced the class of inverse Starlike functions

$g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in A$ which satisfies

$$\mathrm{Re}\left(\frac{zf(z)}{2\int_0^z f(z)dz}\right) > 0,\ z \in E \quad \text{i.e.}\ \frac{zf(z)}{2\int_0^z f(z)dz} < \frac{1+z}{1-z}$$

Gandhi et al.[11]and Rathore et al.[2]established a new class of analytic functions with Fekete-szego inequality using subordination method.
We introduce the class $\mathcal{A}(\alpha,\beta)$ of functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ with satisfying the condition

$$x\left[\frac{\left\{z\left\{z\{zf'(z)\}'\right\}'\right\}'}{f'(z)}\right] + y\left[\frac{\{zf'(z)\}'}{f'(z)}\right] \prec \left(\frac{1+z}{1-z}\right) \quad (6)$$

Let $\mathcal{A}(\alpha,\beta;A,B)$ denotes the subclass of $\mathcal{A}(\alpha,\beta)$ consisting of the functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ satisfying the condition

$$x\left[\frac{\left\{z\left\{z\{zf'(z)\}'\right\}'\right\}'}{f'(z)}\right] + y\left[\frac{\{zf'(z)\}'}{f'(z)}\right] ; -1 \le B \le A \le 1$$
(7)

Let $\mathcal{A}(\alpha,\beta;\delta)$ denotes the subclass of $\mathcal{A}(\alpha,\beta)$ consisting of the functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ with satisfying the condition

$$x\left[\frac{\left\{z\left\{z\{zf'(z)\}'\right\}'\right\}'}{f'(z)}\right] + y\left[\frac{\{zf'(z)\}'}{f'(z)}\right] \prec \left(\frac{1+z}{1-z}\right)^{\lambda} \quad ; \quad \lambda > 0 \quad (8)$$

Let $\mathcal{A}(\alpha,\beta;A,B,\delta)$ denotes the subclass of $\mathcal{A}(\alpha,\beta)$ consisting of the functions $g(z) = z + \sum_{n=2}^{\infty} b_n z^n \in \mathcal{A}$ with satisfying the condition

$$x\left[\frac{\left\{z\left\{z\{zf'(z)\}'\right\}'\right\}'}{f'(z)}\right] + y\left[\frac{\{zf'(z)\}'}{f'(z)}\right] \prec \left(\frac{1+Az}{1+Bz}\right)^{\lambda} - 1 \le B \le A \le 1, \lambda > 0 \quad (9)$$

Here, Symbol $\prec$ stands for subordination.

Principle of Subordination : If $f(z)$ and $F(z)$ are two functions which are analytic in $\mathbb{E}$, then $f(z)$ is called a subordinate to $F(z)$ in $\mathbb{E}$, if there exists a function $w(z)$ which is analytic in $\mathbb{E}$ satisfying the conditions

(i) $w(0) = 0$ and (ii) $|w(z)| < 1$

such that $f(z) = F\big(w(z)\big)$, where $z \in \mathbb{E}$ and we denote it as $f(z) \prec F(z)$. Let $\mathcal{U}$ denote the class of analytic bounded functions of the form

$\boldsymbol{w(z) = \sum_{n=1}^{\infty} d_n z^n}, \boldsymbol{w(0) = 0}, |\boldsymbol{w(z)}| < 1$

Having the restrictions $|\boldsymbol{d_1}| \leq \boldsymbol{1}, |\boldsymbol{d_2}| \leq \boldsymbol{1} - |\boldsymbol{d_1}|^{\boldsymbol{2}}$.

2. Main Results

THEOREM 1. : Prove that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{4(7x+y+2)}{3(17x+2y)(7x+y)} - \frac{\mu}{(7x+y)^2} & , if \mu \leq \frac{4(7x+y)}{\mathbf{3}(17x+2y)} & (10) \\ \frac{2}{3(17x+2y)} & , if \frac{4(7x+y)}{\mathbf{3}(17x+2y)} \leq \mu \leq \frac{4(7x+y+1)(7x+y)}{3(17x+2y)} & (11) \\ \frac{\mu}{(7x+y)^2} - \frac{4(7x+y+2)}{3(17x+2y)(7x+y)} & , if \mu \geq \frac{4(7x+y+1)(7x+y)}{3(17x+2y)} & (12) \end{cases}$$

the results aresharp.

Proof:

On Expanding (6) we have

$$x + y + 2(7x + y)a_2 z + (51xa_3 + 6ya_3 - 28xa_2^2 - 4ya_2^2)z^2 \prec 1 + 2c_1 z + 2(c_2 + c_1^2)z^2 + \ldots\ldots \qquad (13)$$

After identifying the terms in (13), we have

$$|a_3 - \mu a_2^2| \leq \left| \frac{1}{(51x+6y)} \left\{ 2c_2 + 2c_1^2 + \frac{4c_1^2}{(7x+y)} \right\} - \mu \frac{c_1^2}{(7x+y)^2} \right|$$

This leads to

$$|a_3 - \mu a_2^2| \leq \frac{2}{3(17x+2y)} + \left[\left| \frac{2(7x+y+2)}{3(17x+2y)(7x+y)} - \frac{\mu}{(7x+y)^2} \right| - \frac{2}{3(17x+2y)} \right] |c_1|^2 \qquad (14)$$

Case I :when , $\mu \leq \frac{2(7x+y+2)(7x+y)}{3(17x+2y)}$

$$|a_3 - \mu a_2^2| \leq \frac{2}{3(17x+2y)} + \left[\left\{ \frac{2(7x+y+2)}{3(17x+2y)(7x+y)} - \frac{2}{3(17x+2y)} \right\} - \frac{\mu}{(7x+y)^2} \right] |c_1|^2$$

$$|a_3 - \mu a_2^2| \leq \frac{2}{3(17x+2y)} + \left| \left[\frac{4}{3(17x+2y)(7x+y)} - \frac{\mu}{(7x+y)^2} \right] \right| |c_1|^2 \qquad (15)$$

Subcase I(a) : when , $\mu \leq \frac{4(7x+y)}{3(17x+2y)}$

$$|a_3 - \mu a_2^2| \leq \frac{4(7x+y+1)}{3(17x+2y)(7x+y)} - \frac{\mu}{(7x+y)^2} \quad (16)$$

Subcase I(b) : when , $\mu \geq \frac{4(7x+y)}{3(17x+2y)}$

$$|a_3 - \mu a_2^2| \leq \frac{2}{3(17x+2y)} \quad (17)$$

Case II :when , $\mu \geq \frac{2(7x+y+2)(7x+y)}{3(17x+2y)}$

$$|a_3 - \mu a_2^2| \leq \frac{2}{3(17x+2y)} + \left[\left|\frac{\mu}{(7x+y)^2} - \frac{4(7x+y+1)}{3(17x+2y)(7x+y)}\right|\right] \quad (18)$$

Subcase II(a) : when , $\mu \leq \frac{4(7x+y+1)(7x+y)}{3(17x+2y)}$

$$|a_3 - \mu a_2^2| \leq \frac{2}{3(17x+2y)} \quad (19)$$

Subcase II(b) : when , $\mu \geq \frac{4(7x+y+1)(7x+y)}{3(17x+2y)}$

$$|a_3 - \mu a_2^2| \leq \frac{\mu}{(7x+y)^2} - \frac{4(7x+y+2)}{3(17x+2y)(7x+y)} \quad (20)$$

Combining subcase II(a) and subcase I(b), we get

$$|a_3 - \mu a_2^2| \leq \frac{2}{3(17x+2y)} \quad (21)$$

iff

$$\frac{4(7x+y)}{3(17x+2y)} \leq \mu \leq \frac{4(7x+y+1)(7x+y)}{3(17x+2y)}$$

Extremal function

Extreme value for first and third function is $z\{1 + pz\}^q$

where $p = \frac{(51x+6y)-4(7x+y)(7x+y+2)}{(51x+6y)(7x+y)}$ and $q = \frac{(51x+6y)}{(51x+6y)-4(7x+y)(7x+y+2)}$

Extreme value for second function is $\frac{z}{(1-z^2)^p}$

where $p = \frac{2}{(51x+6y)}$

THEOREM 2. : Prove that

$|a_3 - \mu a_2^2| \leq$

$$\begin{cases} \frac{A-B}{(51x+6y)}\left[\frac{(A-B)-B(7x+y)}{(7x+y)}\right]-\frac{(A-B)^2\mu}{4(7x+y)^2}, & if \mu \leq 4\left[\frac{(A-B)-B(7x+y)-7x-y}{(A-B)(51x+6y)}\right](7x+y) \quad (22) \\ \frac{A-B}{(51x+6y)}, & if 4\left[\frac{(A-B)-B(7x+y)-7x-y}{(A-B)(51x+6y)}\right](7x+y) \leq \mu \leq 4\left[\frac{(A-B)-B(7x+y)+7x+y}{(A-B)(51x+6y)}\right](7x+y) \quad (23) \\ \frac{(A-B)^2\mu}{4(7x+y)^2}-\frac{A-B}{(51x+6y)}\left[\frac{(A-B)-B(7x+y)}{(7x+y)}\right], & if \mu \geq 4\left[\frac{(A-B)-B(7x+y)+7x+y}{(A-B)(51x+6y)}\right](7x+y) \quad (24) \end{cases}$$

the results are sharp.

Proof:

On Expanding (7) we have

$$x+y+2(7x+y)a_2z+(51xa_3+6ya_3-28xa_2^2-4ya_2^2)z^2 \prec 1+(A-B)c_1z+(A-B)(c_2-Bc_1^2)z^2+\dots \quad (25)$$

After identifying the terms in (25), we have

$$|a_3 - \mu a_2^2| \leq \left[\left|\frac{(A-B)(c_2-Bc_1^2)}{(51x+6y)}+\frac{(A-B)^2c_1^2}{(51x+6y)(7x+y)}-\frac{(A-B)^2\mu}{4(7x+y)^2}\right|\right]|c_1|^2$$

This leads to

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{(51x+6y)}+\left[\left|\frac{(A-B)^2}{(51x+6y)(7x+y)}-\frac{B(A-B)}{(51x+6y)}-\frac{(A-B)^2\mu}{4(7x+y)^2}\right|-\frac{A-B}{(51x+6y)}\right]|c_1|^2 \quad (26)$$

Case I :when , $\mu \leq 4\left[\frac{(A-B)-B(7x+y)}{(A-B)(51x+6y)}\right](7x+y)$

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{(51x+6y)}+\left[\left|\frac{A-B}{(51x+6y)}\left\{\frac{(A-B)-B(7x+y)-7x-y}{(7x+y)}\right\}-\frac{(A-B)^2\mu}{4(7x+y)^2}\right|\right]|c_1|^2 \quad (27)$$

Subcase I(a) : when , $\mu \leq 4\left[\frac{(A-B)-B(7x+y)-7x-y}{(A-B)(51x+6y)}\right](7x+y)$

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{(51x+6y)}\left[\frac{(A-B)-B(7x+y)}{(7x+y)}\right]-\frac{(A-B)^2\mu}{4(7x+y)^2} \quad (28)$$

Subcase I(b) : when , $\mu \geq 4\left[\frac{(A-B)-B(7x+y)-7x-y}{(A-B)(51x+6y)}\right](7x+y)$

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{(51x+6y)} \quad (29)$$

Case II :when , $\mu \geq 4\left[\frac{(A-B)-B(7x+y)}{(A-B)(51x+6y)}\right](7x+y)$

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{(51x+6y)} + \left[\left|\frac{(A-B)^2\mu}{4(7x+y)^2} - \frac{A-B}{(51x+6y)}\left\{\frac{(A-B)-B(7x+y)+7x+y}{(7x+y)}\right\}\right|\right] |c_1|^2 \quad (30)$$

Subcase II(a) : when , $\mu \leq 4\left[\frac{(A-B)-B(7x+y)+7x+y}{(A-B)(51x+6y)}\right](7x+y)$

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{(51x+6y)} \quad (31)$$

Subcase II(b) : when , $\mu \geq 4\left[\frac{(A-B)-B(7x+y)+7x+y}{(A-B)(51x+6y)}\right](7x+y)$

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)^2\mu}{4(7x+y)^2} - \frac{A-B}{(51x+6y)}\left[\frac{(A-B)-B(7x+y)}{(7x+y)}\right] \quad (32)$$

Combining subcase II(a) and subcase I(b), we get

$$|a_3 - \mu a_2^2| \leq \frac{A-B}{(51x+6y)} \quad (33)$$

iff,

$$4\left[\frac{(A-B)-B(7x+y)-7x-y}{(A-B)(51x+6y)}\right](7x+y) \leq \mu \leq 4\left[\frac{(A-B)-B(7x+y)+7x+y}{(A-B)(51x+6y)}\right](7x+y)$$

Extremal function

Extreme value for first and third function is

$$\{1+pz\}^q$$

where p = $\frac{(A-B)(51x+6y)-8(A-B)(7x+y)+8B(7x+y)^2}{2(51x+6y)(7x+y)}$

$$q = \frac{(A-B)(51x+6y)}{(A-B)(51x+6y)-8(A-B)(7x+y)+8B(7x+y)^2}$$

Extreme value for second function is $\frac{z}{(1-z^2)^p}$

where p = $\frac{A-B}{(51x+6y)}$

Singularities:

Special cases on (33) when A≠B

i) If $A>0, B>0$ then this inequality is holds only for A>B.

ii) If $A>0, B<0$ then this inequality is holds for all values of A and B.

iii)If $A<0, B<0$ then this inequality is holds only for B>A.

iv) If $A<0, B>0$ then this case does not valid.

THEOREM 3. : Prove that

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2\lambda(7x+y+2\lambda)}{(51x+6y)(7x+y)} - \mu\frac{\lambda^2}{(7x+y)^2} & , if\, \mu \leq \frac{4(7x+y)}{(51x+6y)} \quad (34) \\ \frac{\lambda}{3(4x+y)} & , iff\, \frac{4(7x+y)}{(51x+6y)} \leq \mu \leq \frac{4(7x+y+\lambda)(7x+y)}{\lambda(51x+6y)} \quad (35) \\ \mu\frac{\lambda^2}{(7x+y)^2} - \frac{2\lambda(7x+y+2\lambda)}{(51x+6y)(7x+y)} & , if\, \mu \geq \frac{4(7x+y+\lambda)(7x+y)}{\lambda(51x+6y)} \quad (36) \end{cases}$$

the results are sharp.

Proof:

On Eppanding (8) we have

$$x + y + 2(7x+y)a_2 z + (51xa_3 + 6ya_3 - 28xa_2^2 - 4ya_2^2)z^2 \prec 1 + 2\lambda c_1 z + 2\lambda(c_2 + \lambda c_1^2)z^2 + \ldots \quad (37)$$

After identifying the terms in (37), we have

$$|a_3 - \mu a_2^2| \leq \left| \frac{1}{(51x+6y)}\left\{2\lambda c_2 + 2\lambda^2 c_1^2 + \frac{4\lambda^2 c_1^2}{(7x+y)}\right\} - \mu\frac{\lambda^2 c_1^2}{(7x+y)^2} \right|$$

This leads to

$$|a_3 - \mu a_2^2| \leq \frac{2\lambda}{(51x+6y)} + \left[\left|\frac{2\lambda(7x+y+2\lambda)}{(51x+6y)(7x+y)} - \mu\frac{\lambda^2}{(7x+y)^2}\right| - \frac{2\lambda}{(51x+6y)}\right]|c_1|^2 \quad (38)$$

Case I :when , $\mu \leq \frac{2(7x+y+2\lambda)(7x+y)}{\lambda(51x+6y)}$

$$|a_3 - \mu a_2^2| \leq \frac{2\lambda}{(51x+6y)} + \left[\left|\frac{4\lambda^2}{(51x+6y)(7x+y)} - \mu\frac{\lambda^2}{(7x+y)^2}\right|\right] |c_1|^2 \quad (39)$$

Subcase I(a) : when , $\mu \leq \frac{4(7x+y)}{(51x+6y)}$

$$|a_3 - \mu a_2^2| \leq \left[\frac{2\lambda(7x+y+2\lambda)}{(51x+6y)(7x+y)} - \mu\frac{\lambda^2}{(7x+y)^2}\right] \quad (40)$$

Subcase I(b) : when , $\mu \geq \frac{4(7x+y)}{(51x+6y)}$

$$|a_3 - \mu a_2^2| \leq \frac{2\lambda}{(51x+6y)} \quad (41)$$

Case II :when , $\mu \geq \frac{2(7x+y+2\lambda)(7x+y)}{\lambda(51x+6y)}$

$$|a_3 - \mu a_2^2| \leq \frac{2\lambda}{(51x+6y)} + \left[\mu\frac{\lambda^2}{(7x+y)^2} - \frac{4\lambda(7x+y+\lambda)}{(51x+6y)(7x+y)}\right]|c_1|^2$$

Subcase II(a) : when , $\mu \leq \frac{4(7x+y+\lambda)(7x+y)}{\lambda(51x+6y)}$ (42)

$$|a_3 - \mu a_2^2| \leq \frac{2\lambda}{(51x+6y)} \quad (43)$$

Subcase II(b) : when , $\mu \geq \frac{4(7x+y+\lambda)(7x+y)}{\lambda(51x+6y)}$

$$|a_3 - \mu a_2^2| \leq \mu\frac{\lambda^2}{(7x+y)^2} - \frac{2\lambda(7x+y+2\lambda)}{(51x+6y)(7x+y)} \quad (44)$$

Combining subcase II(a) and subcase I(b), we get

$$|a_3 - \mu a_2^2| \leq \frac{2\lambda}{(51x+6y)} \quad (45)$$

$$,iff \ \frac{4(7x+y)}{(51x+6y)} \leq \mu \leq \frac{4(7x+y+\lambda)(7x+y)}{\lambda(51x+6y)}$$

Extremal function

Extreme value for first and third function is $z\{1+pz\}^q$ (46)

where $p = \frac{\lambda(51x+6y)-4(7x+y)(7x+y+2)}{(51x+6y)(7x+y)}$ and $q = \frac{\lambda(51x+6y)}{\lambda(51x+6y)-4(7x+y)(7x+y+2)}$

Extreme value for second function is $\frac{z}{(1-z^2)^p}$ (47)

where $p = \frac{2\lambda}{(51x+6y)}$

ISSN NO:1934-719710

Singularities:

Special cases on (45)

1. If $\lambda > 0$ then the result is hold for all values of λ.
2. If $\lambda < 0$then the result is not valid.

Hence only (1) case is applicable on this theorem.

THEOREM 4. : Prove that

$$|a_3 - \mu a_2^2| \leq$$

$$\begin{cases} \frac{(A-B)\lambda}{(51x+6y)}\left[\frac{(A-B)\lambda-B(7x+y)}{(7x+y)}\right] - \frac{(A-B)^2\mu\lambda^2}{4(7x+y)^2} & , if \mu \leq 4\left[\frac{(A-B)\lambda-B(7x+y)-7x-y}{\lambda(A-B)(51x+6y)}\right](7x+y) \quad (48) \\ \frac{(A-B)\lambda}{(51x+6y)} & , iff\ 4\left[\frac{(A-B)\lambda-B(7x+y)-7x-y}{\lambda(A-B)(51x+6y)}\right](7x+y) \leq \mu \leq 4\left[\frac{(A-B)\lambda-B(7x+y)+7x+y}{\lambda(A-B)(51x+6y)}\right](7x+y) \quad (49) \\ \frac{(A-B)^2\mu\lambda^2}{4(7x+y)^2} - \frac{(A-B)\lambda}{(51x+6y)}\left[\frac{(A-B)\lambda-B(7x+y)}{(7x+y)}\right] & , if \mu \geq 4\left[\frac{(A-B)\lambda-B(7x+y)+7x+y}{\lambda(A-B)(51x+6y)}\right](7x+y) \quad (50) \end{cases}$$

the results are sharp.

Proof:

On Expanding (9) we have

$$x + y + 2(7x+y)a_2 z + (51xa_3 + 6ya_3 - 28xa_2^2 - 4ya_2^2)z^2 \prec 1 + (A-B)c_1\lambda z + (A-B)\lambda(c_2 - B\lambda c_1^2)z^2 + \ldots \quad (51)$$

After identifying the terms in (51), we have

$$|a_3 - \mu a_2^2| \leq \left[\left|\left\{\frac{(A-B)\lambda}{(51x+6y)}(c_2 - B\lambda c_1^2) + \frac{(A-B)^2\lambda^2 c_1^2}{(51x+6y)(7x+y)}\right\} - \frac{(A-B)^2\mu\lambda^2}{4(7x+y)^2}c_1^2\right|\right]$$

This leads to

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\lambda}{(51x+6y)} + \left[\left|\frac{(A-B)^2\lambda^2}{(51x+6y)(7x+y)} - \frac{B(A-B)\lambda}{(51x+6y)} - \frac{(A-B)^2\mu\lambda^2}{4(7x+y)^2}\right| - \frac{(A-B)\lambda}{(51x+6y)}\right]|c_1|^2 \quad (52)$$

Case I :when , $\mu \leq 4\left[\frac{(A-B)\lambda-B(7x+y)}{\lambda(A-B)(51x+6y)}\right](7x+y)$

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\lambda}{(51x+6y)} + \frac{(A-B)\lambda}{(51x+6y)}\left[\frac{(A-B)\lambda-B(7x+y)-7x-y}{(7x+y)}\right] - \frac{(A-B)^2\mu\lambda^2}{4(7x+y)^2} \quad (53)$$

ISSN NO:1934-719711

Subcase I(a) : when , $\mu \leq 4\left[\frac{(A-B)\lambda - B(7x+y)-7x-y}{\lambda(A-B)(51x+6y)}\right](7x+y)$

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\lambda}{(51x+6y)}\left[\frac{(A-B)\lambda - B(7x+y)}{(7x+y)}\right] - \frac{(A-B)^2\mu\lambda^2}{4(7x+y)^2} \quad (54)$$

Subcase I(b) : when , $\mu \geq 4\left[\frac{(A-B)\lambda - B(7x+y)-7x-y}{\lambda(A-B)(51x+6y)}\right](7x+y)$

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\lambda}{(51x+6y)} \quad (55)$$

Case II: when ,$\mu \geq 4\left[\frac{(A-B)\lambda - B(7x+y)}{\lambda(A-B)(51x+6y)}\right](7x+y)$

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\lambda}{(51x+6y)} + \frac{(A-B)^2\mu\lambda^2}{4(7x+y)^2} - \frac{(A-B)\lambda}{(51x+6y)}\left\{\frac{(A-B)\lambda - B(7x+y)+7x+y}{(7x+y)}\right\} \quad (56)$$

Subcase II(a) : when , $\mu \leq 4\left[\frac{(A-B)\lambda - B(7x+y)+7x+y}{\lambda(A-B)(51x+6y)}\right](7x+y)$

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\lambda}{(51x+6y)} \quad (57)$$

Subcase II(b) : when , $\mu \geq 4\left[\frac{(A-B)\lambda - B(7x+y)+7x+y}{\lambda(A-B)(51x+6y)}\right](3x+y)$

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)^2\mu\lambda^2}{4(7x+y)^2} - \frac{(A-B)\lambda}{(51x+6y)}\left[\frac{(A-B)\lambda - B(7x+y)}{(7x+y)}\right] \quad (58)$$

Combining subcase II(a) and subcase I(b), we get

$$|a_3 - \mu a_2^2| \leq \frac{(A-B)\lambda}{(51x+6y)} \quad (59)$$

iff,

$$4\left[\frac{(A-B)\lambda - B(7x+y)-7x-y}{\lambda(A-B)(51x+6y)}\right](7x+y) \leq \mu \leq 4\left[\frac{(A-B)\lambda - B(7x+y)+7x+y}{\lambda(A-B)(51x+6y)}\right](7x+y)$$

Extremal function

Extreme value for first and third function is

$$z\{1+\mathrm{p}z\}^{\mathrm{q}} \qquad (60)$$

where p = $\frac{(A-B)\lambda(51\mathrm{x}+6\mathrm{y})-8\lambda(A-B)(7x+y)+8\mathrm{B}(7x+y)^2}{2(51\mathrm{x}+6\mathrm{y})(7x+y)}$

$$\mathrm{q} = \frac{(A-B)\eta(51\mathrm{x}+6\mathrm{y})}{(A-B)\lambda(51\mathrm{x}+6\mathrm{y})-8\lambda(A-B)(7x+y)+8\mathrm{B}(7x+y)^2}$$

Extreme value for second function is $\frac{z}{(1-z^2)^{\mathrm{p}}}$ (61)

where p = $\frac{(A-B)\lambda}{(51x+6y)}$

Singularities:

Special cases on (59) when A≠B

i) In the case of A>0, $B>0$, $\lambda>0$ or A<0, $B<0$, $\lambda<0$ then ,this inequality holds good only for A>B.

ii) In the case of A>0, $B>0$, $\lambda<0$ or A<0, $B<0$, $\lambda>0$then, this inequality holds good only for B >A.

iii) In the case of A>0, $B<0$, $\lambda<0$ or A<0, $B>0$, $\lambda>0$ then, this inequality does not hold for all values of A and B.

iv) In the case of A>0, $B<0$, $\lambda>0$ or A<0, $B>0$, $\eta<\lambda$ then, this inequality holds good for all values of A and B.

3. Concluding Remarks

If we take A = 1 and B = -1 in the result of theorem 2 , we get the result of theorem 1, therefore our result for the theorem 2 reduces to the result of the theorem1. Hence theorem 2 is the generalization of theorem 1. And the results are sharp and also if we put A = 1 and B = -1 in extremal function of theorem 2, we get the extremal function of theorem 1.

Similarly if we take A = 1 and B = -1 in the result of theorem 4 , we get the result of theorem 3, therefore our result for the theorem 4 reduces to the result of the theorem 3. Hence theorem 4 is the generalization of theorem 3. And the results are sharp and also if we put A = 1 and B = -1 in extremal function of theorem 4, we get the extremal function of theorem 3.

The extremal function given by [(46) and (47)] increases as δ increases and decreases as δ decreases respectively and the extremal function given by [(60) and (61)] also increases and decreases as δ increases and decreases respectively. Hence extremal function is an increasing function.

Acknowledgements: The authors are thankful to referee for valuable comments and suggestions.

4. References

[1] Al-Shaqsi K. and M. Darus, (2008).”On Fekete-Szegö problems for certain subclass of analyticFunctions” Applied Mathematical Sciences, vol. 2 no. 9-12,pp. 431-441.

[2] G. S.Rathore, Gurmeet Singh,Lokendra Kumawat, S. K.Gandhi,Preeri Kumawat (2019)”Some subclasses of a new class of analytic functions under Fekete-Szegö inequality” International journal of Research in Advent Technology, vol. 7, No. 1.

[3] Gurmeet Singh,M. S. Saroa and B. S. Mehrok(2013)”Fekete-szegö inequality for a new class of analytic functions”, Elsevier, Proc. of International conference on Information and Mathematical Sciences, 90-93.

[4] H. R. Abdel-Gawad and D.K. Thomas (1992)”The Fekete-Szegö problem for Strongly close-to-convex functions” Proceedings of the Amercian Mathematical society, vol. 114, no.2 ,345-349.

[5] K.Löwner,(1934).”Uber monotone Matrix funktionen”,Math.Z 38,177-216.

[6] K.O. Babalola.(2009).”The fifth and Sixth coefficient of α– close- to –convex Function”,Kragujevac J. Math. 32, 5-12.

[7] L. Bieberbach, (1916).”Uber einige Extremal Probleme imGebiete der Konformen Abbildung” Math.77, 153-172.

[8] L.Bieberbach, (1916).Uberdie Koeffizientem derjenigem Potenzreihen, welche eineSchlithe Abbildung des Einheitskrises Vermittelen,Preuss. AKad.Wiss Sitzungsb. 940-955.

[9] M. H. Al-Abbadi and M.Darus (2011).Fekete-Sezegö theorem for a certain class of analytic functions,Sanis Malaysiana, vol. 40, no. 4, 385-389.

[10] M. Fekete, and G. Szego, (1933).”Eine Bemerkung Uber ungerade Schlichte Funktionen”, J. London Math. Soc. 8, 85-89.

[11] P. N. Chhichra, (1977).”New Subclasses of the class of close to convex functions”,Procedure of American Mathematical Society ,62, 37-43.

[12] R.M. Goel and B. S. Mehrok(1982)”A Subclass of Univalent functions”, Houston Journal of Mathematics 8, 343-357.

[13] R.M. Goel and B. S. Mehrok.(1990).”A coefficient inequality for certain classes of analytic function”,Tamkang Journal of Mathematics 22, 153-163.

[14] S. K Gandhi, Gurmeet Singh, Preeti Kumawat, G. S. Rathore, Lokendra Kumawat(2018). “A new class of analytic functions with Fekete-Szegö inequality Using subordination Method, International journal of Research in Advent Technology”, vol. 6, No. 9.

[15] S. R. Keogh, and E. R Merkes(1969)"A coefficient inequality for certain classes of analytic Functions", Proceedings of the American Mathematical Society. Vol. 20, 8-12.

[16] S. R. Keogh, and E. R Merkes (1989)."A coefficient inequality for certain classes of analytic Functions", Procedure of American Mathematical Society, 20, 8-12.

[17] T.Hayami, S. Owa, and H. M.Srivastava(2007)."Coefficient inequalities for certain class of analytic and univalent functions", J. Ineq. Pure and Appl. Math.8(4).

[18] W. Kaplan, ,(1952)."Close –to-convex schlicht functions", Michigan Mathematical Journal 1,169-185.